

ON THE SORMANI-WENGER INTRINSIC FLAT CONVERGENCE OF ALEXANDROV SPACES

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ABSTRACT. We study noncollapsing sequences of integral current spaces (X_j, d_j, T_j) with no boundary such that (X_j, d_j) are Alexandrov spaces with nonnegative curvature and diameter uniformly bounded from above and such that the integral current structure T_j has weight 1. We prove that for such sequences, the Gromov-Hausdorff and Sormani-Wenger Intrinsic Flat limits agree.

INTRODUCTION

Burago, Gromov and Perelman proved that sequences of Alexandrov spaces with curvature uniformly bounded from below, diameter and dimension uniformly bounded from above, have subsequences which converge in the Gromov-Hausdorff sense to an Alexandrov space with the same curvature and diameter bounds. The properties of Alexandrov spaces and the Gromov-Hausdorff limit spaces of sequences of these spaces have been amply studied by Alexander-Bishop [1], Alexander-Kapovitch-Petrinin [2], Burago-Gromov-Perelman [5], Burago-Burago-Ivanov [4], Li-Rong [9], Otsu-Shioya [10], and many others.

Sormani-Wenger's intrinsic flat distance between integral current spaces is defined in imitation of Gromov's intrinsic Hausdorff distance (Gromov-Hausdorff distance):

$$(0.1) \quad d_{GH}(X_1, X_2) = \inf d_H^Z(\varphi_1(X_1), \varphi_2(X_2))$$

where d_H^Z denotes the Hausdorff distance and the infimum is taken over all complete metric spaces Z and all distance preserving maps $\varphi_i : X_i \rightarrow Z$, except that the Hausdorff distance d_H is replaced by Federer-Fleming's flat distance d_F [15]. Sormani-Wenger proved that sequences of n -dimensional integral current spaces that are equibounded and have diameter and total mass uniformly bounded from above have subsequences that converge in both Gromov-Hausdorff and intrinsic flat sense. Either the intrinsic flat limit is contained in the Gromov-Hausdorff limit or the intrinsic flat limit is the zero integral current space. See Theorem 4.3 within. For example, a sequence of collapsing tori converges in Gromov-Hausdorff sense to a circle. Since the Hausdorff dimension of the circle does not coincide with the Hausdorff dimension of the tori, the intrinsic flat limit is the zero integral current space. Sormani-Wenger also proved that the Gromov-Hausdorff and intrinsic flat limits agree for sequences of n dimensional Riemannian manifolds that have nonnegative Ricci curvature and volume uniformly bounded from below and from above by positive constants. The intrinsic flat distance has also recently been studied by Lakzian [7], Munn [8], Portegies [12] and the second author [11].

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Before stating our theorem we recall that the notions of countably \mathcal{H}^n rectifiable sets, integral currents and flat convergence on Euclidean space were first introduced in work by Federer-Flemming. Then the notion of integral current was extended by Ambrosio-Kirchheim to complete metric spaces [3]. Sormani-Wenger [15] defined integer rectifiable current spaces, (Y, d, T) which are oriented countably \mathcal{H}^n rectifiable metric spaces (Y, d) with an integer current structure T such that $Y = \{y \in \bar{Y} : \liminf_{r \rightarrow 0} \|T\|(B(y, r))/r^n > 0\}$ where $\|T\|$ is the mass measure of T . Note that integer rectifiable current, T , is determined by the orientation and chart structure of (Y, d) and an integer valued Borel weight function which in turn determines the mass measure $\|T\|$. Integral current spaces are integer rectifiable current spaces whose boundaries are also integer rectifiable. In this paper we assume the Borel weight function determining the integral current structure is 1 and there is no boundary.

Theorem 0.1. *Let (X_j, d_j, T_j) be n -dimensional integral current spaces with weight one and no boundary. Suppose that (X_j, d_j) are Alexandrov spaces with nonnegative curvature and $\text{diam}(X) \leq D$. Then either the sequence converges to the zero integral current space in the intrinsic flat sense*

$$(0.2) \quad (X_j, d_j, T_j) \xrightarrow{\mathcal{F}} \mathbf{0}.$$

or a subsequence converges in the Gromov-Hausdorff sense and intrinsic flat sense to the same space:

$$(0.3) \quad (X_{j_k}, d_{j_k}) \xrightarrow{GH} (X, d)$$

and

$$(0.4) \quad (X_{j_k}, d_{j_k}, T_{j_k}) \xrightarrow{\mathcal{F}} (X, d, T).$$

The proof of Theorem 0.1 is an adaptation to Alexandrov spaces of Sormani-Wenger's proof of the same result for Riemannian manifolds with nonnegative Ricci curvature. From one of their theorems, Theorem 4.3, we obtain a sequence that converges in Gromov-Hausdorff sense to (X, d) and intrinsic flat sense to (Y, d_Y, T) . If the intrinsic flat limit is non zero then X has non zero n -Hausdorff measure by Ambrosio-Kirchheim's characterization of the mass measure, Lemma 3.11. We prove that the set of regular points of X , $R(X)$, is contained in Y in the following way. For a sequence of points x_j that converges to $x \in R(X)$, from Burago-Gromov-Perelman's result (cf. Theorem 2.7) we obtain a sequence of bi-Lipschitz maps from $B_r(x_j)$ to $W_j \subset \mathbb{R}^n$ with Lipschitz constants close to one. The existence of these maps provide an estimate on the intrinsic flat distance between the currents defined on $B_r(x_j)$ and W_j (cf. Sormani-Wenger's lemma, Lemma 4.7). From these estimates we obtain bounds of the form $\|T\|(B_r(x)) \geq C(x, r)r^n$ such that $\lim_{r \rightarrow 0} C(x, r) > 0$. This shows $x \in Y$ and thus $R(x) \subset Y$. To see that $X \subset Y$ we apply Otsu-Shioya's result about the density of the regular set and the Hausdorff dimension of the singular set of X , Ambrosio-Kirchheim's characterization of the mass measure, and the Bishop-Gromov Volume Comparison Theorem for Alexandrov spaces.

In Section 1 we give the definition of Alexandrov space and state Bishop-Gromov Volume Comparison for these spaces. In Section 2 we review Gromov-Hausdorff convergence of sequences of Alexandrov spaces. Then in Section 3 we define integral current spaces. We cover intrinsic flat distance and some important results in Section 4. In Section 5, we calculate upper estimates for the mass of the currents and then prove Theorem 0.1.

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1. ALEXANDROV SPACES

For equivalent definitions and further information see Alexandrov Geometry [2] and A Course in Metric Geometry [4].

Let \mathbb{M}_κ^n denote the n -dimensional complete simply connected space of constant sectional curvature equal to κ . Given three points a, b, c in a length metric space X the triangle $\Delta_\kappa \tilde{a}\tilde{b}\tilde{c} \subset \mathbb{M}_\kappa^n$ that satisfies

$$(1.1) \quad d(a, b) = d(\tilde{a}, \tilde{b}) \quad d(a, c) = d(\tilde{a}, \tilde{c}) \quad \text{and} \quad d(b, c) = d(\tilde{b}, \tilde{c})$$

is called comparison triangle. We denote by $\tilde{\angle}_\kappa \left(a^b \right)_c$ the angle of $\Delta_\kappa \tilde{a}\tilde{b}\tilde{c}$ at \tilde{a} .

Definition 1.1 (From [5]). *A complete length metric space X is said to be an Alexandrov space of curvature greater or equal than κ if for any $x \in X$ there exists an open neighborhood U_x such that for any four points $p, a, b, c \in U_x$ the quadruple condition holds. Namely,*

$$\tilde{\angle}_\kappa \left(p^a \right)_b + \tilde{\angle}_\kappa \left(p^a \right)_c + \tilde{\angle}_\kappa \left(p^b \right)_c \leq 2\pi.$$

We denote by $Alex^n(\kappa)$ the class of n -dimensional Alexandrov spaces with curvature bounded from below by κ and set $Alex^n(\kappa, D) := \{X \in Alex^n(\kappa) : \text{diam}(X) \leq D\}$.

Bishop-Gromov Volume Comparison also holds for Alexandrov spaces.

Theorem 1.2 (Theorem 10.6.6 in [4]). *Let X be an n -dimensional Alexandrov space of curvature $\geq \kappa$. Let V_r^κ denote the volume of a ball of radius r in \mathbb{M}_κ^n . Then for all $x \in X$ the function:*

$$(1.2) \quad r \longmapsto \frac{\mathcal{H}^n(B_r(x))}{V_r^\kappa}$$

is non-increasing.

2. GROMOV-HAUSDORFF CONVERGENCE OF ALEXANDROV SPACES

Given a complete metric space Z , the Hausdorff distance between two subsets A and B of Z is given by

$$(2.1) \quad d_H^Z(A, B) = \inf\{\varepsilon > 0 : A \subset T_\varepsilon(B) \text{ and } B \subset T_\varepsilon(A)\},$$

where $T_\varepsilon(A)$ ($T_\varepsilon(B)$) denotes the ε -tubular neighborhood of A (B).

Hausdorff distance was generalized to metric spaces (X_i, d_i) .

Definition 2.1 (Gromov). *Let (X_i, d_{X_i}) , $i = 1, 2$, be two metric spaces. The Gromov-Hausdorff distance between them is defined as*

$$(2.2) \quad d_{GH}(X, Y) = \inf d_H^Z(\varphi_1(X_1), \varphi_2(X_2))$$

where infimum is taken over all complete metric spaces Z and all isometric embeddings $\varphi_i : X_i \rightarrow Z$.

The function d_{GH} is symmetric and satisfies the triangle inequality. It becomes a distance when restricted to compact metric spaces. The following compactness theorem is due to Gromov.

Theorem 2.2 (Gromov). *Let (X_j, d_j) be a sequence of compact metric spaces. If there exist D , and $N : (0, \infty) \rightarrow \mathbb{N}$ such that for all j*

$$(2.3) \quad \text{diam}(X_j) \leq D$$

and for all ε there are $N(\varepsilon)$ ε -balls that cover X_j , then

$$(2.4) \quad (X_{j_k}, d_{j_k}) \xrightarrow{GH} (X, d_X).$$

In [6] Gromov proved that if a sequence of compact metric spaces (X_j, d_j) converges in Gromov-Hausdorff sense to X_∞ then there is a compact metric space Z and isometric embeddings $\varphi_j : X_j \rightarrow Z$ such that for a subsequence

$$(2.5) \quad d_H^Z(\varphi_j(X_j), \varphi_\infty(X_\infty)) \rightarrow 0.$$

Thus we say that a sequence of points $x_j \in X_j$ converges to $x \in X_\infty$ if $\varphi_j(x_j)$ converges to $\varphi_\infty(x)$ in Z .

Gromov's compactness theorem can be applied to prove convergence of sequences of Alexandrov spaces.

Theorem 2.3 (Burago-Gromov-Perelman 8.5 in [5]). *Let X_j be a sequence of Alexandrov spaces contained in $\text{Alex}^n(\kappa, D)$. Then there is a subsequence X_{j_k} and an Alexandrov space X with curvature $\geq \kappa$, diameter $\leq D$ and Hausdorff dimension $\leq n$ such that*

$$(2.6) \quad X_{j_k} \xrightarrow{GH} X.$$

Theorem 2.4 (Corollary 10.10.11 in [4]). *Let X_j be a sequence of Alexandrov spaces contained in $\text{Alex}^n(\kappa, D)$. Suppose that they converge in Gromov-Hausdorff sense to X . Then $\lim_{j \rightarrow \infty} \mathcal{H}^n(X_j) = 0$ if and only if $\dim_H X < n$.*

The notion of a regular point of an Alexandrov space can be stated in different ways. Here, we present the definition given in terms of Gromov-Hausdorff distance. Let X be an n dimensional Alexandrov space and x a point in X . If for any $\varepsilon > 0$ there is $r > 0$ such that

$$(2.7) \quad d_{GH}(B_r(p), B_r(0, \mathbb{R}^n)) < \varepsilon r,$$

then x is called a regular point of X . We denote by $R(X)$ the set of regular points of X .

Theorem 2.5 (Otsu-Shioya, Theorem A in [10]). *Let X be an n -dimensional Alexandrov space. Then $X \setminus R(X)$ has Hausdorff dimension less or equal than $n - 1$. In particular, $R(X)$ is dense in X .*

Theorem 2.6 is a very important element of the proof of our theorem. So we prove it below.

Theorem 2.6. *For any $\varepsilon > 0$ small, there is $\delta = \delta(n, \kappa, \varepsilon) > 0$ such that for any $X \in \text{Alex}^n(\kappa)$, $p \in X$ and $r > 0$ if*

$$(2.8) \quad d_{GH}(B_r(p), B_r(0, \mathbb{R}^n)) < \delta r,$$

then there exists a bi-Lipschitz map

$$(2.9) \quad f : B_{\delta r}(p) \rightarrow W \subset \mathbb{R}^n$$

with $\text{Lip}(f), \text{Lip}(f^{-1}) \leq 1 + \varepsilon$. In particular,

$$(2.10) \quad B_{(1-\varepsilon)r}(f(p)) \subset f(B_r(p)) \subset B_{(1+\varepsilon)r}(f(p)).$$

In order to prove the theorem we need the following. A collection of points $\{(a_i, b_i)\}_{i=1}^n$ is called an (n, δ) -explosion if for all $i \neq j$ the following holds

$$(2.11) \quad \tilde{z}_\kappa(p_{a_i}^{a_i}) > \frac{\pi}{2} - \delta, \quad \tilde{z}_\kappa(p_{b_j}^{b_i}) > \frac{\pi}{2} - \delta,$$

$$(2.12) \quad \tilde{z}_\kappa(p_{b_i}^{a_i}) > \pi - \delta, \quad \text{and} \quad \tilde{z}_\kappa(p_{b_j}^{a_i}) > \frac{\pi}{2} - \delta.$$

A map $f: X \rightarrow Y$ is said to be a λ -almost isometry if it is bi-Lipschitz with $\text{Lip}(f), \text{Lip}(f^{-1}) \leq 1 + \lambda$.

Theorem 2.7 ([5] Theorem 9.4). *Suppose that $p \in X \in \text{Alex}^n(\kappa)$ has an (n, δ) -explosion (a_i, b_i) . Then the map $f: X \rightarrow \mathbb{R}^n$ given by $f(q) = (|a_1 q|, \dots, |a_n q|)$ maps a small neighbourhood U of the point p , $\tau(\delta, \delta_1)$ -almost isometrically onto a domain in \mathbb{R}^n . Here*

$$\delta_1 = \max_{1 \leq i \leq n} \left\{ \frac{\text{diam}(U)}{|pa_i|}, \frac{\text{diam}(U)}{|pb_i|} \right\};$$

$\tau(\delta, \delta_1) > 0$ depending only on $\delta, \delta_1, n, \kappa$ and $\lim_{\delta, \delta_1 \rightarrow 0} \tau(\delta, \delta_1) = 0$.

Proof of Theorem 2.6. We first show that p has an $(n, \tau(\delta))$ -explosion. Let $\{(\tilde{a}_i, \tilde{b}_i)\}$ be a $(n, 0)$ -explosion for the origin o in \mathbb{R}^n with $d(\tilde{a}_i, o) = d(\tilde{b}_i, o) = r$, that is, $d(\tilde{a}_i, \tilde{b}_i) = 2r$ and $d(\tilde{a}_i, \tilde{a}_j) = d(\tilde{b}_i, \tilde{b}_j) = d(\tilde{a}_i, \tilde{b}_j) = \sqrt{2}r$ for all $i \neq j$.

Let $\phi: B_r(p) \rightarrow B_r(0, \mathbb{R}^n)$ be a δr -Gromov-Hausdorff approximation and $a_i, b_i \in B_r(p)$ such that $\phi(a_i) = \tilde{a}_i, \phi(b_i) = \tilde{b}_i$. For $\delta > 0$ small enough, direct computation shows that $\{(a_i, b_i)\}$ is a $(n, 20\delta)$ -explosion for p in X . For simplicity, we assume $\kappa = 0$ and verify it for $\tilde{z}_0(p_{a_i}^{a_i})$ and $\tilde{z}_0(p_{b_i}^{a_i})$ as follows.

$$\begin{aligned} |\cos(\tilde{z}_0(p_{a_i}^{a_i}))| &= \left| \frac{|pa_i|^2 + |pa_j|^2 - |a_i a_j|^2}{2|pa_i||pa_j|} \right| \\ &\leq \frac{((1+\delta)r)^2 + ((1+\delta)r)^2 - ((\sqrt{2}-\delta)r)^2}{2(1-\delta)^2 r^2} \\ &\leq 10\delta. \end{aligned}$$

$$\begin{aligned} 1 + \cos(\tilde{z}_0(p_{b_i}^{a_i})) &= 1 + \frac{|pa_i|^2 + |pb_i|^2 - |a_i b_i|^2}{2|pa_i||pb_i|} \\ &= \frac{(|pa_i| + |pb_i|)^2 - |a_i b_i|^2}{2|pa_i||pb_i|} \\ &\leq \frac{(2(1+\delta)r)^2 - (2(1-\delta)r)^2}{2(1-\delta)^2 r^2} \\ &\leq 10\delta. \end{aligned}$$

Let $U = B_{\delta r}(p)$ be a neighborhood of p . The map f defined as in Theorem 2.7 maps $B_{\delta r}(p)$, $\tau(\delta, \delta_1)$ -almost isometry to an open domain in \mathbb{R}^n , where $\delta_1 \leq \frac{2\delta r}{(1-\delta)r} \leq 4\delta$. \square

3. INTEGRAL CURRENT SPACES

In this section we review integral current spaces as presented in Section 2 of Sormani-Wenger [15]. We refer to their work for a complete exposition of integral current spaces. For readers interested in integral currents we refer to Ambrosio-Kirchheim's paper [3].

Let Z be a metric space. We denote by $\mathcal{D}^m(Z)$ the collection of $(m+1)$ -tuples of Lipschitz functions such that the first entry function is bounded:

$$(3.1) \quad \mathcal{D}^m(Z) = \{(f, \pi) = (f, \pi_1, \dots, \pi_m) \mid f, \pi_i : Z \rightarrow \mathbb{R} \text{ Lipschitz and } f \text{ is bounded}\}.$$

Definition 3.1 (Ambrosio-Kirchheim). *Let Z be a complete metric space. A multilinear functional $T : \mathcal{D}^m(Z) \rightarrow \mathbb{R}$ is called an m dimensional current if it satisfies:*

- i) *If there is an i such that π_i is constant on a neighborhood of $\{f \neq 0\}$ then $T(f, \pi) = 0$.*
- ii) *T is continuous with respect to the pointwise convergence of the π_i for $\text{Lip}(\pi_i) \leq 1$.*
- iii) *There exists a finite Borel measure μ on Z such that for all $(f, \pi) \in \mathcal{D}^m(Z)$*

$$(3.2) \quad |T(f, \pi)| \leq \prod_{i=1}^m \text{Lip}(\pi_i) \int_Z |f| d\mu.$$

The collection of all m dimensional currents of Z is denoted by $\mathbf{M}_m(Z)$.

Definition 3.2 (Ambrosio-Kirchheim). *Let $T : \mathcal{D}^m(Z) \rightarrow \mathbb{R}$ be an m -dimensional current. The mass measure of T is the smallest Borel measure $\|T\|$ such that (3.2) holds for all $(f, \pi) \in \mathcal{D}^m(Z)$.*

The mass of T is defined as

$$(3.3) \quad M(T) = \|T\|(Z) = \int_Z d\|T\|.$$

The most studied currents come from pushforwards and Example 3.4.

Definition 3.3 (Ambrosio-Kirchheim Defn 2.4). *Let $T \in \mathbf{M}_m(Z)$ and $\varphi : Z \rightarrow Z'$ be a Lipschitz map. The pushforward of T to a current $\varphi_\# T \in \mathbf{M}_m(Z')$ is given by*

$$(3.4) \quad \varphi_\# T(f, \pi) = T(f \circ \varphi, \pi_1 \circ \varphi, \dots, \pi_m \circ \varphi).$$

Example 3.4 (Ambrosio-Kirchheim). *Let $h : A \subset \mathbb{R}^m \rightarrow \mathbb{Z}$ be an L^1 function. Then $\llbracket h \rrbracket : \mathcal{D}^m(\mathbb{R}^m) \rightarrow \mathbb{R}$ given by*

$$(3.5) \quad \llbracket h \rrbracket(f, \pi) = \int_{A \subset \mathbb{R}^m} h f \det(\nabla \pi_i) d\mathcal{L}^m$$

in an m dimensional current.

Example 3.5 (Ambrosio-Kirchheim). *Let $h : A \subset \mathbb{R}^m \rightarrow \mathbb{Z}$ be an L^1 function and $\varphi : \mathbb{R} \rightarrow Z$ be a bilipschitz map, then $\varphi_\# \llbracket h \rrbracket \in \mathbf{M}_m(Z)$. Explicitly,*

$$(3.6) \quad \varphi_\# \llbracket h \rrbracket(f, \pi_1, \dots, \pi_m) = \int_{A \subset \mathbb{R}^m} h(f \circ \varphi) \det(\nabla(\pi_i \circ \varphi)) d\mathcal{L}^m.$$

Remark 3.6. $\nabla \pi_i, \nabla(\pi_i \circ \varphi)$ are defined almost everywhere by Rademacher's Theorem.

Now we are ready to define integer rectifiable currents.

Definition 3.7 (Defn 4.2, Thm 4.5 in Ambrosio-Kirchheim [3]). *Let $T \in \mathbf{M}_m(Z)$. T is an integer rectifiable current if it has a parametrization of the form $(\{\varphi_i\}, \{\theta_i\})$, where*

- i) $\varphi_i : A_i \subset \mathbb{R}^m \rightarrow Z$ is a countable collection of bilipschitz maps such that A_i are precompact Borel measurable with pairwise disjoint images,

ii) $\theta_i \in L^1(A_i, \mathbb{N})$ such that

$$(3.7) \quad T = \sum_{i=1}^{\infty} \varphi_{i\#} \llbracket \theta_i \rrbracket \quad \text{and} \quad \mathbf{M}(T) = \sum_{i=1}^{\infty} \mathbf{M}(\varphi_{i\#} \llbracket \theta_i \rrbracket).$$

The mass measure is

$$(3.8) \quad \|T\| = \sum_{i=1}^{\infty} \|\varphi_{i\#} \llbracket \theta_i \rrbracket\|.$$

The space of m dimensional integer rectifiable currents on Z is denoted by $\mathcal{I}_m(Z)$.

We define a boundary operator and the canonical set of a current.

Definition 3.8 (Ambrosio-Kirchheim). *An integral current is an integer rectifiable current, $T \in \mathcal{I}_m(Z)$, such that ∂T defined as*

$$(3.9) \quad \partial T(f, \pi_1, \dots, \pi_{m-1}) = T(1, f, \pi_1, \dots, \pi_{m-1})$$

is a current. $\mathbf{I}_m(Z)$ denote the space of m dimensional integral currents on Z .

Definition 3.9 (Ambrosio-Kirchheim). *Let $T \in \mathbf{M}_m(Z)$, the canonical set of T is the set*

$$(3.10) \quad \text{set}(T) = \{p \in Z : \Theta_{*m}(\|T\|, p) > 0\},$$

where $\Theta_{*m}(\|T\|, p)$ is the $\|T\|$ lower density of x .

$$(3.11) \quad \Theta_{*m}(\|T\|, p) = \liminf_{r \rightarrow 0} \frac{\|T\|(B_r(p))}{\omega_m r^m}$$

Here ω_m denotes the volume of the unit ball in \mathbb{R}^m .

Definition 3.10 (Sormani-Wenger). *Let (X, d) be a metric space and $T \in \mathbf{I}_m(\overline{X})$. If $\text{set}(T) = X$ then (X, d, T) is called an m dimensional integral current space.*

We denote by \mathcal{M}^m the space of m dimensional integral current spaces and by \mathcal{M}_0^m the space of m dimensional integral current spaces whose canonical set is precompact.

Ambrosio-Kirchheim proved in [3] that if $T \in \mathcal{I}_m(Z)$ then $\text{set}(T)$ is a countably \mathcal{H}^m rectifiable metric space. Hence, m dimensional integral current spaces (X, d, T) satisfy the property of X being countably \mathcal{H}^m rectifiable. Ambrosio-Kirchheim also characterized the mass measure.

Lemma 3.11 (Ambrosio-Kirchheim). *Let $T \in \mathcal{I}_m(Z)$ with parametrization $(\{\varphi_i\}, \theta_i)$. Then there is a function*

$$(3.12) \quad \lambda : \text{set}(T) \rightarrow [m^{-m/2}, 2^m/\omega_m]$$

such that

$$(3.13) \quad \Theta_{*m}(\|T\|, x) = \theta_T(x) \lambda(x)$$

for \mathcal{H}^m almost every $x \in \text{set}(T)$ and

$$(3.14) \quad \|T\| = \theta_T \lambda \mathcal{H}^m \llcorner \text{set}(T),$$

where ω_m denotes the volume of an unitary ball in \mathbb{R}^m and $\theta_T : Z \rightarrow \mathbb{N} \cup \{0\}$ is given by

$$(3.15) \quad \theta_T = \sum_{i=1}^{\infty} \theta_i \circ \varphi_i^{-1} 1_{\varphi_i(A_i)}.$$

4. INTRINSIC FLAT CONVERGENCE

In this section we give the definition of intrinsic flat distance between integral current spaces and state various intrinsic flat convergence theorems.

Definition 4.1 (Sormani-Wenger). *Let $(X_i, d_i, T_i) \in \mathcal{M}^m$. Then,*

$$(4.1) \quad d_{\mathcal{F}}((X_1, d_1, T_1), (X_2, d_2, T_2)) = \inf\{\mathbf{M}(U) + \mathbf{M}(V)\},$$

where the infimum is taken over all complete metric spaces, (Z, d) , and all integral currents, $U \in \mathbf{I}_m(Z)$, $V \in \mathbf{I}_{m+1}(Z)$, for which there exists isometric embeddings

$$(4.2) \quad \varphi_i : (\bar{X}_i, d_i) \rightarrow (Z, d)$$

with

$$(4.3) \quad \varphi_{1\#}T_1 - \varphi_{2\#}T_2 = U + \partial V.$$

We include the zero integral current space denoted by $\mathbf{0}$ to \mathcal{M}^m . The $\mathbf{0}$ m dimensional integral current isometrically embeds into any Z with $\varphi_{\#}\mathbf{0} = \mathbf{0} \in \mathbf{I}_m(Z)$.

If one of the integral currents is the zero integral current, say $(X_2, d_2, T_2) = \mathbf{0}$, we can choose $V = \mathbf{0}$ and $U = (X_1, d_1, T_1)$. Then

$$(4.4) \quad d_{\mathcal{F}}((X_1, d_1, T_1), \mathbf{0}) \leq \mathbf{M}(T_1).$$

This estimate with X_1 equal to a ball and T_1 an integral current structure defined on X_1 will be used in the proof of Theorem 0.1 to calculate the lower density of points in the Gromov-Hausdorff limit space.

In Theorem 3.27 of [15] is proven that $d_{\mathcal{F}}$ is a distance on the class of precompact integral currents, \mathcal{M}_0^m .

Remark 4.2 (Sormani-Wenger). *The intrinsic flat distance between (X_1, d_1, T_1) and (X_2, d_2, T_2) can be written using the Flat norm as*

$$(4.5) \quad d_{\mathcal{F}}(M_1, M_2) = \inf d_F^Z(\varphi_{1\#}T_1, \varphi_{2\#}T_2),$$

where the infimum is taken over the same spaces, embeddings and currents as in Definition 4.1.

Theorem 4.3 (Sormani-Wenger). *Let (X_j, d_j, T_j) be a sequence of m dimensional integral current spaces. If there exist D, M and $N : (0, \infty) \rightarrow \mathbb{N}$ such that for all j*

$$(4.6) \quad \text{diam}(X_j) \leq D, \quad \mathbf{M}(T_j) + \mathbf{M}(\partial T_j) \leq M$$

and, for all ε there are $N(\varepsilon)$ ε -balls that cover X_j , then

$$(4.7) \quad (X_{j_k}, d_{j_k}) \xrightarrow{GH} (X, d_X) \quad \text{and} \quad (X_{j_k}, d_{j_k}, T_{j_k}) \xrightarrow{\mathcal{F}} (Y, d, T),$$

where either (Y, d, T) is an m dimensional integral current space with $Y \subset X$ or it is the $\mathbf{0}$ current space.

Remark 4.4. *When we apply Theorem 4.3 we generally consider that all the spaces are embedded in a complete metric space. This assumption comes from the proof of the theorem. Sormani-Wenger show that there exist a complete metric space Z and isometric embeddings $\varphi_j : X_j \rightarrow Z$, $\varphi : X \rightarrow Z$ such that for the a subsequence*

$$(4.8) \quad d_H^Z(\varphi_j(X_j), \varphi(X)) \rightarrow 0$$

and

$$(4.9) \quad d_F^Z(\varphi_{j\#}T_j, \varphi_{\#}T) \rightarrow 0.$$

Definition 4.5. Let (X_j, d_j, T_j) be a sequence of integral current spaces such that

$$(4.10) \quad (X_j, d_j, T_j) \xrightarrow{\mathcal{F}} (X_\infty, d_\infty, T_\infty).$$

A sequence $x_j \in X_j$ is called Cauchy if there exist a complete metric space Z and isometric embeddings $\varphi_j : X_j \rightarrow Z$, $\varphi : X \rightarrow Z$ such that

$$(4.11) \quad d_H^Z(\varphi_j(X_j), \varphi(X)) \rightarrow 0$$

and $\varphi_j(x_j)$ converges in Z (not necessarily to a point in $\varphi(\bar{X})$).

Given $T \in \mathbf{M}_m(Z)$ and A a Borel set, the restriction of T to A is a current $T \llcorner A \in \mathbf{M}_m(Z)$ given by

$$(4.12) \quad (T \llcorner A)(f, \pi) = T(f\chi_A, \pi).$$

where χ_A is the indicator function of A . This definition is part of a broader definition of restriction. See Definition 2.19 in [15]. The mass of $T \llcorner A$ is $\|T\|(A)$.

In the proof of our theorem we will calculate the lower density of points using the following.

Theorem 4.6 (Sormani [13]). Let (X_j, d_j, T_j) be a sequence of integral current spaces such that

$$(4.13) \quad (X_j, d_j, T_j) \xrightarrow{\mathcal{F}} (X_\infty, d_\infty, T_\infty)$$

and $x_j \in X_j$ a Cauchy sequence. Then for almost all $r > 0$

$$(4.14) \quad M_j(r) = (B_r(x_j), d_j, T_j \llcorner B_r(x_j)).$$

are integral current spaces and

$$(4.15) \quad M_j(r) \xrightarrow{\mathcal{F}} M_\infty(r).$$

Lemma 4.7 (Sormani-Wenger). Let (X_i, d_i) be complete metric spaces and $\varphi : X_1 \rightarrow X_2$ be a $\lambda > 1$ biLipschitz map. If $T \in \mathbf{I}_m(X_1)$, then for $M_1 = (\text{set}(T), d_1, T)$ and $M_2 = (\text{set}(\varphi_\# T), d_2, \varphi_\# T)$

$$(4.16) \quad d_{\mathcal{F}}(M_1, M_2) \leq k_{\lambda, m} \max\{\text{diam}(\text{spt } T), \text{diam}(\varphi(\text{spt } T))\} \mathbf{M}(T)$$

where $k_{\lambda, m} = \frac{1}{2}(m+1)\lambda^{m-1}(\lambda-1)$.

5. PROOF OF THE MAIN THEOREM

In order to apply Sormani-Wenger's theorem, Theorem 4.3, in the proof of the main theorem we need the next lemma.

Lemma 5.1. Let $(X, d) \in \text{Alex}^n(\kappa, D)$ equipped with an integral current structure T with weight equal to one. Then

$$(5.1) \quad \|T\|(X) \leq C(n, \kappa, D).$$

Moreover,

$$(5.2) \quad \|T\|(B_r(x)) \leq c(n, \kappa)r^n$$

for all r .

Proof. By the Bishop-Gromov Volume Comparison for Alexandrov spaces (Theorem 10.2 in [5]) we know that

$$(5.3) \quad \mathcal{H}^n(X) \leq \mathcal{H}^n(B_D(\mathbb{M}_\kappa^n)) = c(n, \kappa, D),$$

where \mathbb{M}_κ^n denotes the n -dimensional complete simply connected space of constant sectional curvature κ and $B_D(\mathbb{M}_\kappa^n)$ a ball in \mathbb{M}_κ^n with radius D . By Lemma 3.11,

$$(5.4) \quad \|T\|(X) = \int_X \theta_T(x) \lambda(x) d\mathcal{H}^n \leq 2^n / \omega_n \mathcal{H}^n(X) \leq 2^n / \omega_n c(n, \kappa, D),$$

where we used that $\theta_T = 1$ and $\lambda \leq 2^n / \omega_n$.

The second part is proved in a similar way. By Bishop-Gromov Volume Comparison for Alexandrov spaces we have that

$$(5.5) \quad \mathcal{H}^n(B_r(x)) \leq \mathcal{H}^n(B_r(\mathbb{M}_\kappa^n)) = c(n, \kappa) r^n.$$

Thus,

$$(5.6) \quad \|T\|(B_r(x)) = \int_{B_r(x)} \theta_T(x) \lambda(x) d\mathcal{H}^n \leq 2^n / \omega_n \mathcal{H}^n(B_r(x)) \leq 2^n / \omega_n c(n, \kappa) r^n.$$

□

Proof of Theorem 0.1. By the compactness theorem for Alexandrov spaces, Theorem 2.3, there is an Alexandrov space (X, d_X) with nonnegative curvature, $\text{diam}(X) \leq D$ and Hausdorff dimension $\leq n$ and, a convergent subsequence:

$$(5.7) \quad (X_{j_k}, d_{j_k}) \xrightarrow{\text{GH}} (X, d_X).$$

This, together with the uniform bound of $\|T_j\|$ given by Lemma 5.1 allow us to apply Theorem 4.3. Thus, possibly taking a further subsequence there is an intrinsic flat convergent subsequence:

$$(5.8) \quad (X_{j_k}, d_{j_k}, T_{j_k}) \xrightarrow{\mathcal{F}} (Y, d_Y, T).$$

where either (Y, d_Y, T) is the zero integral current space or (Y, d_Y) can be viewed as a subspace of (X, d_X) . To simplify notation we suppose that

$$(5.9) \quad (X_j, d_j) \xrightarrow{\text{GH}} (X, d_X) \text{ and } (X_j, d_j, T_j) \xrightarrow{\mathcal{F}} (Y, d_Y, T).$$

If (Y, d_Y, T) is the zero integral current space then there is nothing else to prove. Otherwise, $Y \subset X$. We have to show that $X \subseteq Y$. Recall that

$$(5.10) \quad Y = \{x \in \bar{Y} : \liminf_{r \rightarrow 0} \frac{\|T\|(B_r(x))}{r^n} > 0\}$$

where $\|T\|$ is the mass measure of T defined in \bar{Y} .

We first prove that the set of regular points of X is contained in Y , $R(X) \subset Y$. We will prove our claim by showing that for $x \in R(X)$ there is a sequence $x_j \in X_j$ that converges to x , $r_0 > 0$ and a function $C(n, r)$ that satisfies $\lim_{r \rightarrow 0} C(n, r) > 0$ such that

$$(5.11) \quad \|T\|(B_{\delta r}(x)) > C(n, r)(\delta r)^n$$

for almost all $r \leq r_0$. More explicitly, we use the fact that

$$(5.12) \quad \|T\|(B_{\delta r}(x)) \geq d_{\mathcal{F}}((B_{\delta r}(x), d, T \llcorner B_{\delta r}(x)), \mathbf{0}) = \lim_{j \rightarrow \infty} d_{\mathcal{F}}((B_{\delta r}(x_j), d_j, T_j \llcorner B_{\delta r}(x_j)), \mathbf{0})$$

and obtain an estimate of the form

$$(5.13) \quad d_{\mathcal{F}}((B_{\delta r}(x_j), d_j, T_j \llcorner B_{\delta r}(x_j)), \mathbf{0}) > C(n, r)(\delta r)^n$$

for almost all $r \leq r_0$.

Let $\varepsilon > 0$ and take δ as in Theorem 2.6. Since (Y, d_Y, T) is the nonzero integral current space it follows that $\mathcal{H}^n(X) > 0$. This is proven using the characterization of the mass measure (Lemma 3.11). Hence, by Otsu-Shiroya's theorem, Theorem 2.5, $R(X)$ is non-empty. Then for each $x \in R(X)$ there exists $r_0 > 0$ such that

$$(5.14) \quad d_{GH}(B_{r_0}(x), B_{r_0}(0, \mathbb{R}^n)) < \delta r_0/2,$$

where $B_{r_0}(0, \mathbb{R}^n)$ is the ball with radius r_0 centered at 0 in the Euclidean space \mathbb{R}^n .

Since x is a point in the Gromov-Hausdorff limit space X there is a subsequence $x_j \in X_j$ that converges to x . From the Gromov-Hausdorff convergence of the sequence of spaces (X_j, d_j) and the convergence of the sequence of points x_j it follows that for j large enough

$$(5.15) \quad d_{GH}(B_{r_0}(x_j), B_{r_0}(x)) < \delta r_0/2.$$

Using the triangle inequality we get that for j sufficiently large

$$(5.16) \quad d_{GH}(B_{r_0}(x_j), B_{r_0}(0, \mathbb{R}^n)) < \delta r_0.$$

Then, by Theorem 2.6 and our choice of δ we have bi-Lipschitz maps

$$(5.17) \quad f_j: B_{\delta r_0}(x_j) \rightarrow W_j \subset \mathbb{R}^n$$

such that $\text{Lip}(f_j), \text{Lip}(f_j^{-1}) \leq 1 + \varepsilon$.

For each $r \leq r_0$ we can restrict the previous maps and obtain bi-Lipschitz maps

$$(5.18) \quad f_j: B_{\delta r}(x_j) \rightarrow f_j(B_{\delta r}(x_j)) \subset \mathbb{R}^n$$

such that $\text{Lip}(f_j), \text{Lip}(f_j^{-1}) \leq 1 + \varepsilon$. Then we define:

$$(5.19) \quad M_j(r) = (B_{\delta r}(x_j), d_j, T_j \llcorner B_{\delta r}(x_j))$$

$$(5.20) \quad M'_j(r) = (f_j(B_{\delta r}(x_j)), d_{\mathbb{R}^n}, f_{j\#}(T_j \llcorner B_{\delta r}(x_j))).$$

We get an estimate of the distance between the integral currents $M_j(r)$ and $M'_j(r)$ applying Lemma 4.7,

$$(5.21) \quad d_{\mathcal{F}}(M_j(r), M'_j(r)) \leq k_{1+\varepsilon, n} \max\{\text{diam}(\text{spt } T_j), \text{diam}(f_j(\text{spt } T_j))\} \mathbf{M}(T_j \llcorner B_{\delta r_0}(x_j))$$

$$(5.22) \quad \leq \frac{1}{2}(n+1)(1+\varepsilon)^n \varepsilon \cdot 2(1+\varepsilon)\delta r \cdot c(n, 0)(\delta r)^n$$

$$(5.23) \quad = c(n, 0)(n+1)(1+\varepsilon)^{n+1} \varepsilon (\delta r)(\delta r)^n.$$

For $\varepsilon < 1$, the ball $B_{(1-\varepsilon)\delta r}(f_j(x_j))$ is contained in $f_j(B_{\delta r}(x_j))$. Note that $M'_j(r)$ has the standard current structure since f_j is bilipschitz and $M_j(r)$ has weight one. Then,

$$(5.24) \quad d_{\mathcal{F}}(\mathbf{0}, M'_j(r)) \geq \omega_n((1-\varepsilon)\delta r)^n.$$

Using the triangle inequality,

$$(5.25) \quad d_{\mathcal{F}}(M_j(r), \mathbf{0}) \geq d_{\mathcal{F}}(\mathbf{0}, M'_j(r)) - d_{\mathcal{F}}(M'_j(r), M_j(r))$$

$$(5.26) \quad \geq \omega_n((1-\varepsilon)\delta r)^n - c(n, 0)(n+1)(1+\varepsilon)^{n+1} \varepsilon (\delta r)(\delta r)^n$$

$$(5.27) \quad = \left(\omega_n(1-\varepsilon)^n - c(n, 0)(n+1)(1+\varepsilon)^{n+1} \varepsilon (\delta r) \right) (\delta r)^n.$$

Then for r sufficiently small $C(n, r) = \omega_n(1-\varepsilon)^n - c(n, 0)(n+1)(1+\varepsilon)^{n+1} \varepsilon (\delta r)$ is positive. Moreover, $\lim_{r \rightarrow 0} C(n, r) > 0$.

With the above estimate we show that $\|T\|(B_{\delta r}(x)) > C(n, r)(\delta r)^n$. By Sormani's theorem, Theorem 4.6, $d_{\mathcal{F}}((B_{\delta r}(x_j), d_j, T_j \llcorner B_{\delta r}(x_j)), (B_{\delta r}(x), d, T \llcorner B_{\delta r}(x))) \rightarrow 0$. Then,

$$(5.28) \quad \|T\|(B_{\delta r}(x)) \geq d_{\mathcal{F}}((B_{\delta r}(x), d, T \llcorner B_{\delta r}(x)), \mathbf{0})$$

$$(5.29) \quad = \lim_{j \rightarrow \infty} d_{\mathcal{F}}((B_{\delta r}(x_j), d_j, T_j \llcorner B_{\delta r}(x_j)), \mathbf{0}) > C(n, r)(\delta r)^n.$$

Then,

$$(5.30) \quad \liminf_{r \rightarrow 0} \frac{\|T\|(B_{\delta r}(x))}{(\delta r)^n} > 0.$$

This proves that $x \in Y$. Hence, $R(X) \subset Y$.

To complete the proof, recall that $\|T\|$ can be written as $\|T\| = \theta_T \lambda \mathcal{H}^n \llcorner Y$ where $\theta_T, \lambda : \bar{Y} \rightarrow \mathbb{R}$ are nonnegative integrable functions such that $\theta > 0$ in Y and $\lambda \geq n^{-n/2}$. By Theorem 2.5 we know that $R(X)$ is dense in X and $\mathcal{H}^n(X \setminus R(X)) = 0$. It follows that $R(X)$ is dense in Y and $\mathcal{H}^n(Y \setminus R(X)) = 0$. Thus, for $x \in X$

$$(5.31) \quad \|T\|(B_r(x)) = \int_{B_r(x)} \theta_T(y) \lambda(y) d\mathcal{H}^n \llcorner Y$$

$$(5.32) \quad = \int_{B_r(x) \cap R(X)} \theta_T(y) \lambda(y) d\mathcal{H}^n \llcorner Y$$

$$(5.33) \quad \geq n^{-n/2} \mathcal{H}^n(B_r(x) \cap R(X)) r^n$$

$$(5.34) \quad \geq n^{-n/2} \mathcal{H}^n(X) / D^n r^n$$

where we used $\mathcal{H}^n(B_r(x) \cap R(X)) = \mathcal{H}^n(B_r(x))$ and Bishop-Gromov volume comparison for Alexandrov spaces, Theorem 1.2. It follows that $\liminf_{r \rightarrow 0} \|T\|(B_r(x)) / r^n > 0$. This shows that $X \subset Y$. \square

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